A **mapping** transforms one set of numbers into a different set of numbers. Mappings can be described in words or through an algebraic equation, and some can be represented graphically.

A mapping is a **function** if every **input** has a single distinct **output**. There are two types of function:

One-to-one: Each input maps to a unique output, for example $f(x) = x + 3$

If the output is known, an **inverse** function can be used to find the associated input

Many-to-one: Multiple inputs can map to the same output, for example $f(x) = \sin x$

Here, angles such as 30°, 150° and infinitely many more all map to the same output of 0.5

one-to-one
function

many-to-one
function
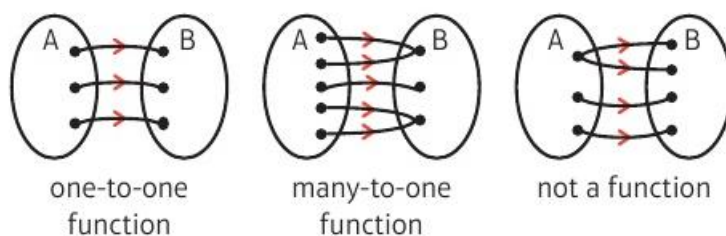
not a function

The **domain** of a function is the set of all possible inputs (x-values if you are graphing the function)

The **range** of a function is the set of all possible outputs (y-values if you are graphing the function)

Many mappings can be made into functions by restricting the domain.

For example, consider $f(x) = \sqrt{x}$. Note that there isn't a $\pm$ sign, so this implies the positive square root is taken.

If we take the domain as $x \in \mathbb{R}$ then $\sqrt{x}$ isn't a function, as inputting negative values of x doesn't give an output.

If we restrict the domain to $x \geq 0$, then $\sqrt{x}$ is a function, as every element in the domain maps to a single value.

The function can be written as $f(x) = \sqrt{x}, x \in \mathbb{R}, x \geq 0$ Alternative notation: $f: x \mapsto \sqrt{x}, x \in \mathbb{R}, x \geq 0$
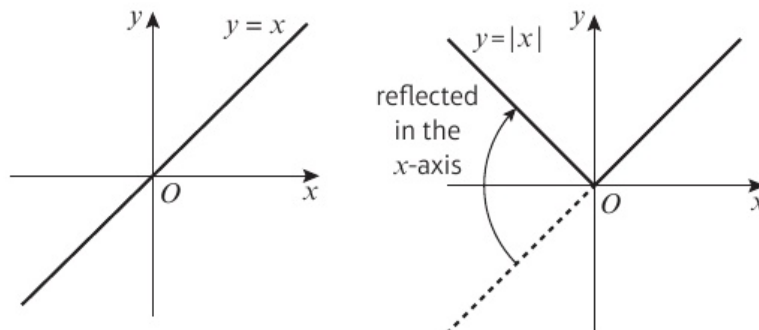
The Modulus Function

The **modulus** of a number a, written as $|a|$, is its **non-negative** numerical value. So, both $|5|$ and $|-5|$ equal 5.

In general, a **modulus function** is a function of the type $y = |f(x)|$, such that

- when $f(x) \geq 0$, $y = f(x)$

- when $f(x) < 0$, $y = -f(x)$

To sketch graph of a modulus function $y = |f(x)|$, start with the graph $y = f(x)$ and reflect any sections below the x-axis in the x-axis (a vertical reflection)

To solve equations with modulus functions, **draw a sketch** to check if solutions lie on the ordinary or reflected lines.

Inequalities of the form $|f(x)| < k$ can be written in the form $-k < f(x) < k$, and vice versa.

Transformation of Graphs

Vertical (outside the function): use BIDMAS   Horizontal (inside the function): use reverse BIDMAS.

Horizontal: $f(x - a) \rightarrow$ translation $\begin{pmatrix} a \\ 0 \end{pmatrix}$ $f(ax) \rightarrow$ stretch, x-direction, scale factor $\frac{1}{a}$

Vertical: $f(x) + a \rightarrow$ translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$ $af(x) \rightarrow$ stretch, y-direction, scale factor a

Reflections: $-f(x) \rightarrow$ vertical reflection (in x-axis) $f(-x) \rightarrow$ horizontal reflection (in y-axis)

Modulus Functions:

$y = |f(x)|$ Sketch $y = f(x)$ and reflect any parts below the x-axis (where $f(x) < 0$) in the x-axis.

$y = f(|x|)$ Sketch $y = f(x)$ for positive x-values (where $x \geq 0$), and reflect this in the y-axis.

Composite Functions

Two or more functions can be combined to make a new function, called a **composite** function.

$fg(x)$ means "apply the function g to the variable x first, and then apply the function f to the result":

$$fg(x) = f(g(x))$$

Algebraically, substitute $g(x)$ for x wherever it appears in the function $f(x)$ and simplify the result if appropriate.

Inverse Functions

For any one-to-one function $f(x)$ there exists an **inverse function $f^{-1}(x)$**

The **inverse** of a function performs the opposite operation(s) to the original function, taking elements from the range of the original function and mapping them back to the domain of the original function. This is why inverse functions only exist for one-to-one functions.

Many-to-one functions can be changed into one-to-one functions by restricting the domain. For example, $\sin x$ is many-to-one but if you restrict the domain to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ the function becomes one-to-one and an inverse function $\sin^{-1} x$ or $\arcsin x$ can be defined:
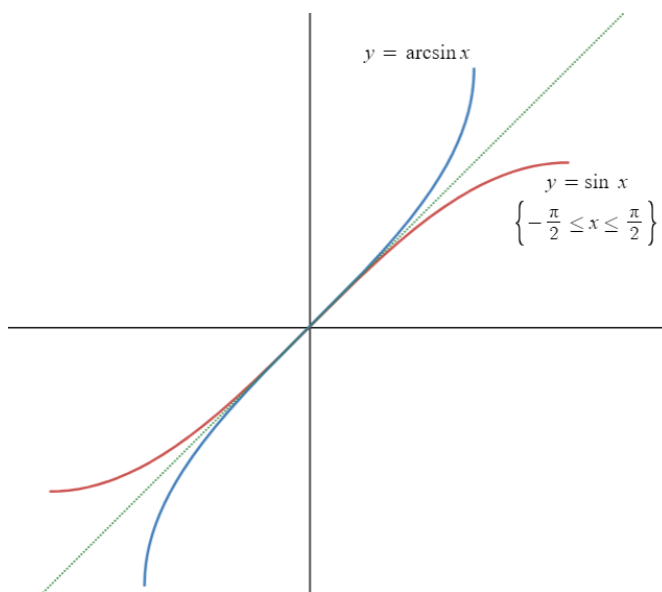
The following properties are always true, and can be seen in the graphs of $y = \sin x$ and $y = \arcsin x$

The **domain** of a function gives the **range** of the inverse

The **range** of a function gives the **domain** of the inverse

$f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$

Example	Domain	Range
$f(x)$	$-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$	$-1 \leq y \leq 1$
$f^{-1}(x)$	$-1 \leq x \leq 1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$

The maximum and minimum values of $f(x)$ correspond to minimum and maximum values of $f^{-1}(x)$ respectively.

To find an inverse function algebraically,

- Start with $y = f(x)$

- Swap the variables, to get $x = f(y)$

- Rearrange to make y the subject, to get $y = f^{-1}(x)$