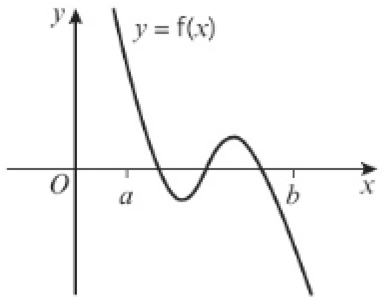
A2 Pure – Chapter 10 – Numerical Methods

Finding Roots using a Change of Sign

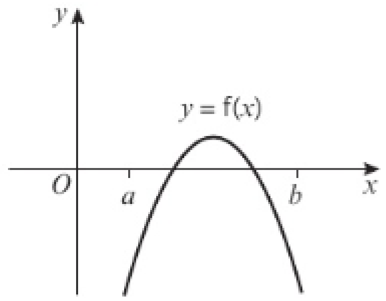
If a function $f(x)$ is continuous (not discontinuities such as asymptotes) on the interval $[a, b]$ *and* $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has *at least one* root, x, which satisfies $a < x < b$.

There are three situations to watch out for when locating roots using a change of sign.

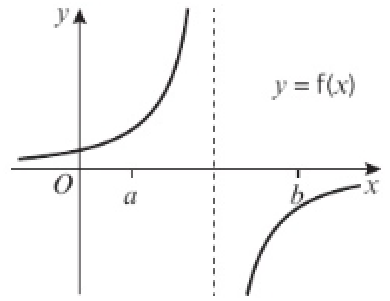
Roots can exist where there isn't a change of sign, and a change of sign doesn't necessarily mean there is a root:



There are multiple roots within the interval $[a, b]$. In this case there is an **odd number** of roots



There are multiple roots within the interval $[a, b]$, but a sign change does not occur. In this case there is an **even number** of roots.



There is a vertical asymptote within interval $[a, b]$. A sign change does occur, but there is no root.

Checking accuracy

If you believe you have found a root to a particular degree of accuracy, you may be asked to verify this.

For example, if you believe that $\alpha = 1.694$ is a solution to $f(\alpha) = 0$ to three decimal places, check either side:

$$f(1.6935) = \cdots$$

$$f(1.6945) = \cdots$$

If there is a change of sign, then $1.6935 < \alpha < 1.6945$, and so $\alpha = 1.694$ to three decimal places.

The Newton-Raphson Method

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

The Newton-Raphson method uses tangent lines to find increasingly accurate approximations of a root.

The value of x_{n+1} is the point at which the tangent to the graph at $(x_n, f(x_n))$ intersects the x-axis.

If the initial value isn't chosen carefully, the method may not converge on the required root, or converge very slowly.

If the initial value is close to a turning point, the tangent will intersect the x-axis a long way from x_0.

If any value x_i in the Newton-Raphson method is **at** a turning point, the method will fail because $f'(x_i) = 0$, and the formula would require a division by zero, which is not valid. Graphically, the tangent line would run parallel to the x-axis and never intersect it.

Iteration

To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formula:

$$\boldsymbol{x_{n+1} = g(x_n)}$$

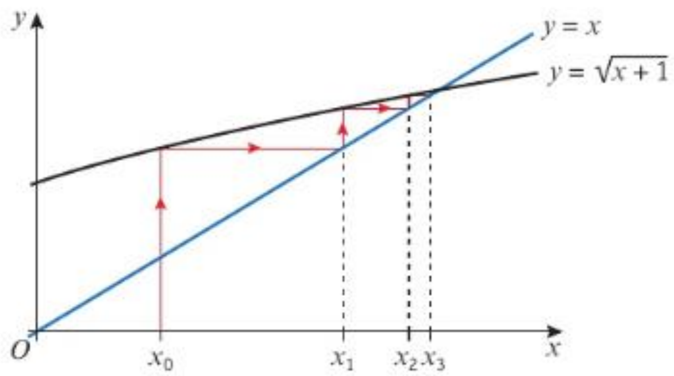
This process finds the intersection of the graphs $y = x$ and $y = g(x)$, which is also the solution to $f(x) = 0$.

Some iterations will converge to a root, and this can happen in two ways.

One way is that successive iterations get closer and closer to the root from the same direction. Graphically these iterations create a series of steps, known as a **staircase** diagram.

This example shows a solution to $x^2 - x - 1 = 0$, using the iteration formula $x_{n+1} = \sqrt{x_n + 1}$ and $x_0 = 0.5$

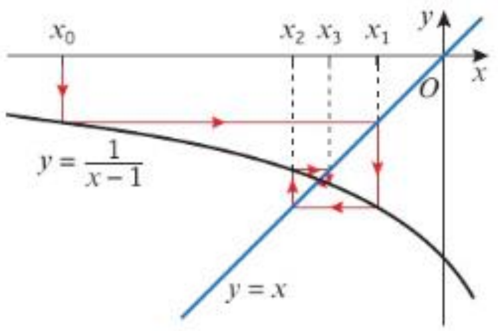
Staircase Diagram:



The other way is that successive iterations alternate either side of the root. Graphically these iterations create a rotating pattern converging on the root, known as a **cobweb** diagram.

This example also shows a solution to $x^2 - x - 1 = 0$, using the iteration formula $x_{n+1} = \frac{1}{x_n - 1}$ and $x_0 = -2$

Cobweb Diagram:



Not all iterations or starting diagrams will find a root!

$x^2 - x - 1 = 0$ can be rearranged to give the iteration formula $x_{n+1} = {x_n}^2 - 1$, but using a starting point of $x_0 = 2$, the iteration **diverges** from the root: