A2 Pure – Chapter 9 – Differentiation

Basic Derivatives

Function	e^x	$\ln x$	$\sin x$	$\cos x$	$\tan x$
Derivative	e^x	$\frac{1}{x}$ or x^{-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Apart from the result for $\tan x$, **THESE ARE NOT GIVEN IN THE FORMULA BOOK**. Learn them!

Differentiating Sine and Cosine from First Principles

You are expected to be able to differentiate $\sin x$ and $\cos x$ from first principles.

To do this, you need to know the small angle approximations: $\boldsymbol{\sin x \approx x}$ $\boldsymbol{\cos x \approx 1 - \frac{1}{2}x^2}$

You will also need the compound angle formulae:

$$\boldsymbol{\sin(A+B) = \sin A \cos B + \cos A \sin B} \qquad \boldsymbol{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

The expression for the gradient of a small chord on a graph $y = f(x)$ is: gradient $= \frac{f(x+h) - f(x)}{h}$

The derivative of the function is the limit of this expression as $h \to 0$: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Using the function $f(x) = \sin x$, we get the following:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$ (using the compound angle formula)

$$f'(x) = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$ (taking a factor of $\sin x$ from the first and last terms)

$$f'(x) = \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h}$$ (splitting into two fractions)

$$f'(x) = \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x$$ (separating x and h terms in each fraction)

We can now use the small angle approximations:

As $h \to 0$, $\frac{\sin h}{h} \approx \frac{h}{h} = 1$

As $h \to 0$, $\frac{\cos h - 1}{h} \approx \frac{\left(1 - \frac{1}{2}h^2\right) - 1}{h} = -\frac{\frac{1}{2}h^2}{h} = -\frac{1}{2}h \to 0$

Substituting these into the previous result, we get $f'(x) = (0)\sin x + (1)\cos x$

$$f'(x) = \cos x$$

Hence the derivative of $\sin x$ is $\cos x$, and by a similar method you can also find the derivative of $\cos x$:

$$\boldsymbol{\frac{d}{dx}(\sin x) = \cos x} \qquad \boldsymbol{\frac{d}{dx}(\cos x) = -\sin x}$$

These derivatives are only valid when the angles are given in radians!

We can also generalise these rule using the chain rule (taught later in the unit):

$$\boldsymbol{\frac{d}{dx}(\sin kx) = k\cos kx} \qquad \boldsymbol{\frac{d}{dx}(\cos kx) = -k\sin kx}$$

Differentiating Exponentials and Logarithms

You need to know and be able to use the standard results for differentiating exponentials and logarithms:

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

You don't need to be able to differentiate these from first principles!

There are a couple of derivatives which follow from these, which you may be asked to show.

Differentiating a^x and a^{kx}

Let $y = a^x$

$y = e^{\ln(a^x)}$ (this is true as the exponential cancels the logarithm)

$y = e^{x \ln a}$ (bring the power to the front)

Then $\frac{dy}{dx} = \ln a \times e^{x \ln a}$ (bring the constant to the front when differentiating)

$\frac{dy}{dx} = \ln a \times e^{\ln(a^x)}$ (bring the x into the logarithm as a power)

$\frac{dy}{dx} = a^x \ln a$ (the exponential cancels the logarithm)

Hence

$$\frac{d}{dx}(a^x) = a^x \ln a \quad \text{and} \quad \frac{d}{dx}\left(a^{kx}\right) = ka^{kx} \ln a$$

Differentiating $\ln kx$

Let $y = \ln kx$

$y = \ln k + \ln x$ (laws of logarithms)

Then $\frac{dy}{dx} = \frac{d(\ln k + \ln x)}{dx}$ (obviously)

$\frac{dy}{dx} = \frac{d(\ln x)}{dx}$ (since $\ln k$ is a constant)

$\frac{dy}{dx} = \frac{1}{x}$ (using the standard result)

$$\frac{d}{dx}(\ln kx) = \frac{1}{x}$$

You can also use the chain rule, seen later, to get the same result.

<u>Chain Rule</u> (not given in the formula book)

You can use the chain rule to differentiate composite functions, or functions of functions.

If y is a function of u and u is a function of x, then $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

With practice, you're expected to be able to use the chain rule without setting it out formally.

<u>Differentiating powers of functions using the chain rule</u>

When differentiating a polynomial, we "bring the power to the front and reduce the power by one"

When differentiating a power of a function $f(x)$ by using the chain rule, this becomes:

"Bring the power and the derivative of the function in the bracket to the front and reduce the power by one"

In function notation, this can be generalised as: **If** $\boldsymbol{y = [\,f(x)\,]^{n}\ ,\ \frac{dy}{dx} = n\,f'(x)\,[\,f(x)\,]^{n-1}}$

For example, $y = (3x^2+1)^5 \ \rightarrow\ \frac{dy}{dx} = 5(6x)(3x^2+1)^4 = 30x(3x^2+1)$ as the derivative of $3x^2+1$ is $6x$

<u>Differentiating functions of functions using the chain rule</u>

When differentiating a function of a function $f(x)$ by using the chain rule:

"Differentiate the outer function 'normally' and bring the derivative of the inner function to the front"

In function notation, this can be generalised as: **If** $\boldsymbol{y = f\big(g(x)\big)\ ,\ \frac{dy}{dx} = g'(x)\,f'\big(g(x)\big)}$

For example, $y = \sin(x^3) \ \rightarrow\ \frac{dy}{dx} = 3x^2\cos(x^3)$ as the derivative of x^3 is $3x^2$ and the derivative of $\sin x$ is $\cos x$.

<u>Product Rule</u> (not given in the formula book)

If $y = uv$, where u and v are functions of x, then $$\boldsymbol{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Abbreviate! You may also write this as $$\boldsymbol{\frac{dy}{dx} = uv' + u'v}$$

<u>Quotient Rule</u> (given in the formula book, in function notation)

If $= \frac{u}{v}$, where u and v are functions of x, then $$\boldsymbol{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Abbreviate! You may also write this as $$\boldsymbol{\frac{dy}{dx} = \frac{u'v - uv'}{v^2}}$$

Implicit Differentiation

An equation of the form $y = f(x)$ is given **explicitly**.

An equation involving functions of x and y such as $2x + xy = y^2$ is given **implicitly**.

Implicit equations can be difficult, sometimes impossible, to rearrange into an explicit form $y = f(x)$ so you can differentiate them. Even if it is possible, the function $f(x)$ may not be easy to differentiate.

Instead, we can differentiate implicitly. To differentiate implicitly with respect to x:

- Differentiate terms in x only with respect to x as normal
- Differentiate terms In y only with respect to y and multiply by $\frac{dy}{dx}$ (this uses the chain rule)
- Differentiate terms in both x and y by treating them as the product of an x term and a y term and using the product rule. Don't forget to include $\frac{dy}{dx}$ when differentiating the term in y in the product rule!
- The result can then be rearranged to make $\frac{dy}{dx}$ the subject.

Generally, your expression for $\frac{dy}{dx}$ will be in terms of both x and y. This is fine!

Here's a basic example: *Differentiate $x^3 + 2xy = 3y^2$ with respect to x*

x^3 differentiates to $3x^2$ and $3y^2$ differentiates to $6y\frac{dy}{dx}$

$2xy$ can be split into $2x$ and y and differentiated using the product rule (using x and $2y$ would give the same result):

$u = 2x \quad u' = 2 \quad v = y \quad v' = 1\frac{dy}{dx}$ so $\frac{d}{dx}(2xy) = 2x\frac{dy}{dx} + 2y$

Combining these, we get $3x^2 + 2x\frac{dy}{dx} + 2y = 6y\frac{dy}{dx}$

Finally, rearrange to make $\frac{dy}{dx}$ the subject:

$$x^2 + 2y = 6y\frac{dy}{dx} - 2x\frac{dy}{dx}$$

$$x^2 + 2y = \frac{dy}{dx}(6y - 2x)$$

$$\frac{dy}{dx} = \frac{6y - 2x}{x^2 + 2y}$$

Rates of Change and the Chain Rule

You can use the chain rule to connect rates of change in situations involving more than two variables.

For example, say you know the rate at which a circle's radius r is increasing with respect to time t is 5cm per second.

This can be written as $\frac{dr}{dt} = 5$

We can also work out the rate of change $\frac{dA}{dt}$ of the circle's area A

This is because $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ using the chain rule. We're given $\frac{dr}{dt}$ and though we aren't given $\frac{dA}{dr}$ in the question, we know that $A = \pi r^2$ and can differentiate this to get $\frac{dA}{dr} = 2\pi r$. We now have all we need to work out $\frac{dA}{dt}$:

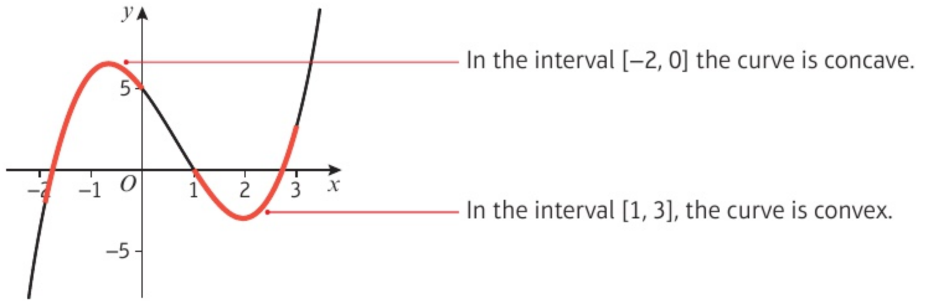
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 5 = 10\pi r$$

Second Derivatives

A function $f(x)$ is **concave** on a given interval if and only if the second derivative $f''(x) \leq 0$ for all x in the interval.

A function $f(x)$ is **convex** on a given interval if and only if the second derivative $f''(x) \geq 0$ for all x in the interval.

A **point of inflection** is a point at which $f''(x)$ changes sign. It is not necessarily a turning point.



Here are some specific types of trigonometric derivatives you may be asked to find.

These are not common exam questions, but they may come up.

Differentiating Trigonometric Functions

You can differentiate $\tan x$, $\sec x$, $\operatorname{cosec} x$ and $\cot x$ by writing them as fractions using $\sin x$ and $\cos x$ and using the quotient rule. The results are given in the formula book, and you can use these unless explicitly asked to show them:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \qquad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Differentiating Inverse Trigonometric Functions

The derivatives of $\arcsin x$, $\arccos x$ and $\arctan x$ are only used in further maths, and the results are given in that section of the formula book:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

However, you could be asked to show these results are true.

For example, to differentiate $\arcsin x$,

Start with: $y = \arcsin x$

Take sine of both sides: $x = \sin y$

Differentiate with respect to y: $\frac{dx}{dy} = \cos y$ (you could also differentiate implicitly here)

Take reciprocals: $\frac{dy}{dx} = \frac{1}{\cos y}$

To rewrite in terms of x (which we know equals $\sin y$), we use a rearrangement of $\sin^2\theta + \cos^2\theta \equiv 1$

Substitute $\cos y = \sqrt{1-\sin^2 y}$: $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

Substitute $x = \sin y$ $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

You can find the derivative of $\arccos x$ in exactly the same way.

For $\arctan x$ we use the same method and the equivalent identity $1 + \tan^2 y \equiv \sec^2 y$ on the denominator.