AS Pure – Chapter 12 – Differentiation

Definitions

The **gradient** of a **curve** at a given point is defined as the gradient of the **tangent** to the curve at that point.

The gradient is also known as the **rate of change** of the curve.

The **gradient function**, or **derivative**, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$ and is defined as:

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

The fraction in the expression is the gradient of a **chord** drawn from the point $(x, f(x))$ to another point slightly further along the curve, $(x+h, f(x+h))$, where $\boldsymbol{h}$ is the (small) increase in the x-coordinate.

$\lim_{h \to 0}$ means "the limit as h tends to zero". The expression can't be evaluated when $h = 0$, but as h gets smaller the expression gets closer to a fixed (or **limiting**) value. h can also be denoted by $\boldsymbol{\delta x}$, meaning "a small change in x".

Using this definition to find the derivative of a particular function is called **differentiating from first principles**.

Finding the Derivative

The gradient function can be used to find the gradient of the curve for any value of x.

For all real values of n, and for a constant a:

- **If $\boldsymbol{f(x) = x^n}$, then $\boldsymbol{f'(x) = nx^{n-1}}$**
- **If $\boldsymbol{y = x^n}$, then $\boldsymbol{\frac{dy}{dx} = nx^{n-1}}$**
- **If $\boldsymbol{f(x) = ax^n}$ then $\boldsymbol{f'(x) = anx^{n-1}}$**
- **If $\boldsymbol{y = ax^n}$ then $\boldsymbol{\frac{dy}{dx} = anx^{n-1}}$**

Constant terms in a function do not contribute to the gradient function.

Individual terms in a function can be differentiated separately.

For example, the quadratic curve with equation $y = ax^2 + bx + c$ has the derivative $\frac{dy}{dx} = 2ax + b$

In general, if $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$

Tangents and Normals

The **normal** to a curve at a particular point is a line drawn perpendicular to the **tangent** at that point.

The **tangent** to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation:

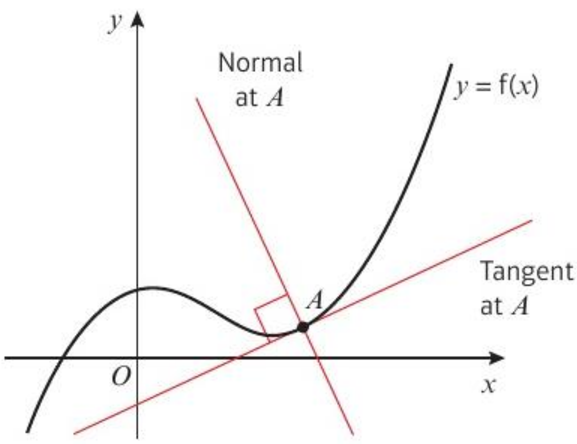
$$y - f(a) = f'(a)(x - a)$$

Note that this is the general formula for a straight line, $y - y_1 = m(x - x_1)$, where m is the value of the derivative.

The **normal** to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation:

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

Here, the gradient is the negative reciprocal of the derivative.



Increasing and Decreasing Functions

A function is **increasing** when the gradient is positive, and **decreasing** when the gradient is negative.

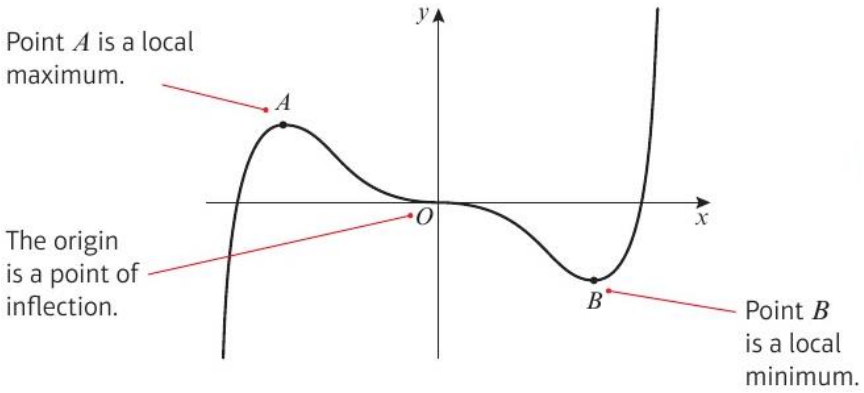
The function $f(x)$ is increasing on the interval $[a, b]$ if $f'(x) > 0$ for all values of x such that $a < x < b$.

The function $f(x)$ is decreasing on the interval $[a, b]$ if $f'(x) < 0$ for all values of x such that $a < x < b$.

Stationary Points

A **stationary point** (or turning point) on a curve is any point where the curve has a **gradient of zero**.

The diagram below illustrates three types of stationary point:



You can determine whether a stationary point is a **local maximum**, a **local minimum** or a **point of inflection** by looking at the gradient of the curve on either side. For a small positive value of h:

Gradient before $f'(x-h)$	$f'(x)$	Gradient after $f'(x+h)$	Type of stationary point	Before		After
Positive	Zero	Negative	Local maximum	⋰	⋯	⋱
Negative	Zero	Positive	Local minimum	⋱	⋯	⋰
Negative	Zero	Negative	Point of inflection	⋱	⋯	⋱
Positive	Zero	Positive	Point of inflection	⋰	⋯	⋰

Second Derivatives

A more rigorous way of checking the nature of a stationary point is to calculate the **second derivative**.

Differentiating a function $y = f(x)$ twice gives you the **second order derivative**, $f''(x)$ or $\frac{d^2y}{dx^2}$

This function is the **rate of change of the gradient function**. It tells us how the gradient is changing at a given point.

We are particularly interested how the gradient is changing close to stationary points.

If a function $f(x)$ has a stationary point when $x = a$, then:

- **If $f''(a) > 0$, the point is a local minimum (the gradient is zero but about to increase)**

- **If $f''(a) < 0$, the point is a local maximum (the gradient is zero but about to decrease)**

If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to check the gradient at points on either side to determine its nature.