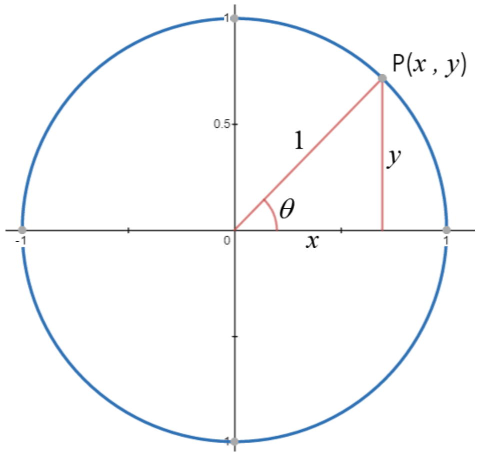
Chapter 10 – Trigonometric Equations

Trigonometry and the unit circle

For a point P on the unit circle such that the line segment OP makes an angle θ with the positive x-axis,

$$\cos\boldsymbol{\theta} = \boldsymbol{x} \qquad \sin\boldsymbol{\theta} = \boldsymbol{y} \qquad \tan\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$



For $0° < \theta < 90°$, this is a simple result of the SOHCAHTOA definitions of the basic trigonometric functions from GCSE.

The unit circle approach allows us to extend our definition for angles up to $360°$. Just remember that the x- and y-coordinates can be negative further round the circle!

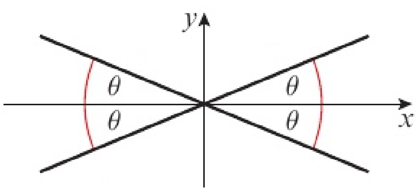
Values of trigonometric functions between 90˚ and 360˚

When you use an inverse trigonometric function, your calculator will only give you the **principal value** for each:

- For $\sin^{-1}\theta$, you will get a value in the range $-90° \leq \theta \leq 90°$
- For $\cos^{-1}\theta$, you will get a value in the range $0° \leq \theta \leq 180°$
- For $\tan^{-1}\theta$, you will get a value in the range $-90 < \theta < 90°$

These inverse trigonometric functions are also referred to as **arcsin**, **arccos** and **arctan**.

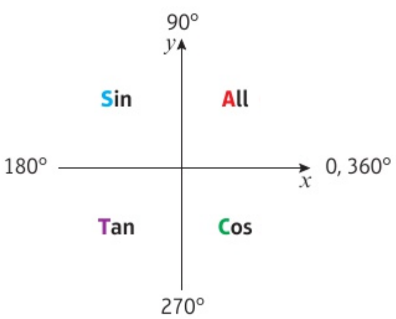
To find trigonometric values for angles between 90˚ and 360˚, you need to find the value for the corresponding acute angle and apply the correct sign based on which quadrant the angle is in. **Draw a sketch to avoid errors!**



The corresponding acute angle for any angle between 90˚ and 360˚ is the angle made with the x-axis, as shown in the diagram.

For example, 155˚, 205˚ and 335˚ all correspond to the acute angle 25˚

The absolute values of sine (and cosine and tangent) for all four angles are the same. However, they may be positive or negative, depending on the quadrant.



The **CAST diagram**, which would more accurately be called the ASTC diagram, is a revision aid to help remember which trigonometric functions are positive and negative in each quadrant.

- 1st quadrant: ALL three trig functions have a positive value
- 2nd quadrant: SINE is the only positive trig function
- 3rd quadrant: TAN is the only positive trig function
- 4th quadrant: COSINE is the only positive trig function

This is a result of the positive and negative values of x and y for a coordinate on the unit circle in each quadrant.

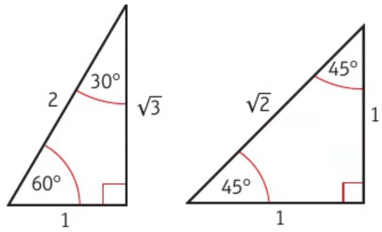
It also means that, since all three basic trig functions are positive in two quadrants and negative in two quadrants, then the inverse trig functions all have **two** solutions between 0 and 360 (with a few exceptions for multiples of 90).

Exact Trigonometric Values

You are still expected to know the exact trigonometric values for 30˚, 45˚ and 60˚.

These can be found using the basic triangles shown to the right (half an equilateral triangle of side length 2 units for 30˚ and 60˚, and half a unit square for 45˚)

You can then use the CAST diagram to find exact values for the trig functions of angles for which 30˚, 45˚ and 60˚ are the corresponding acute angles.



Solving Trigonometric Equations

When solving trigonometric equations, always check the question for the range in which your answers should lie.

For example, you may be asked to solve $\sin\theta = \frac{\sqrt{3}}{2}$ in the range $0° \leq \theta \leq 360°$

In this case, your calculator will only give the value 60˚, and you will need to use your knowledge of the sine curve or the CAST diagram to find the second solution in the range, 120˚

Not all solutions will require you to use the symmetry of the graphs or the CAST diagram. Remember that the sine and cosine graphs simply repeat every 360˚, and the tan graph repeats every 180˚.

When solving equations where the trigonometric function has a horizontal transformation, write down all the required values *immediately* after using the inverse trigonometric function on your calculator.

For example, when solving $\cos 2\theta = \frac{\sqrt{3}}{2}$ in the range $0° \leq \theta \leq 360°$,

$\cos 2\theta = \frac{\sqrt{3}}{2}$

$2\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ My calculator will give one value – I need to work out **all** those required

$2\theta = 30°,\ 330°,\ 390°,\ 690°$ Write all values ***up to 720˚*** using a sketch of the cosine graph if needed

$\theta = 15°,\ 165°,\ 195°,\ 345°$ This way, when you divide by 2 you still have all values up to 360˚

Trigonometric Identities

There are only two trigonometric identities you are required to know for AS Maths (with plenty to come next year!)

The most important trigonometric identity is:

$$\boldsymbol{\sin^2\theta + \cos^2\theta \equiv 1}$$

The "triple equals" sign indicates that this is an **identity**, rather than an equation to be solved.

You can show this is true between 0˚ and 90˚ using SOHCAHTOA and Pythagoras, but it is true for all values of θ

You need to know this by memory, and **you will use this a lot**, especially in Year 13!

The other identity you need is:

$$\boldsymbol{\tan\theta \equiv \frac{\sin\theta}{\cos\theta}}$$

This can also be shown using SOHCAHTOA between 0˚ and 90˚, but it is true for all values except when $\cos\theta = 0$