A2 Pure – Chapter 7 – Trigonometry and Modelling

Compound Angle Formulae

The compound angle formulae (also called addition formulae) are given in the Formula Book:

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Note the use of $\mp$, which indicates that the positive case on the LHS corresponds to the negative case on the RHS.

You may be asked to prove these geometrically (see your examples!), and the result for $\tan(A \pm B)$ can also be derived by dividing $\sin(A \pm B)$ by $\cos(A \pm B)$

Double Angle Formulae

By substituting $A = B$ into these and simplifying, we can easily derive what are known as the double angle formulae:

$$\boldsymbol{\sin 2A \equiv 2\sin A \cos A} \qquad \boldsymbol{\cos 2A \equiv \cos^2 A - \sin^2 A} \qquad \boldsymbol{\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}}$$

THESE ARE NOT GIVEN IN THE FORMULA BOOK!

You are expected to remember these or be able to derive them quickly from the compound angle formulae.

By substituting the identity $\sin^2 A + \cos^2 A = 1$ into the double angle formula for $\cos 2A$, we get these results:

$$\boldsymbol{\cos 2A \equiv 2\cos^2 A - 1 \qquad \Rightarrow \qquad \cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)}$$

$$\boldsymbol{\cos 2A \equiv 1 - 2\sin^2 A \qquad \Rightarrow \qquad \sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)}$$

The form on the left is useful for re-writing equations with a cosine double angle as quadratics in $\sin\theta$ or $\cos\theta$.

The form on the right is used in calculus to allow us to integrate $\sin^2 A$ and $\cos^2 A$.

Using the Compound and Double Angle Formulae

You may need to use these to solve equations or prove other trigonometric identities. Here are a few tips:

- If you forget a double angle formula, look up the compound angle formula and work it out from those.
- If you're solving an equation with a double and single angle, try using a double angle formula. The formula for $\cos 2A$ is particularly useful here for getting a quadratic equation, used with $\sin^2 A + \cos^2 A \equiv 1$
- In fact, if you've got a double angle and you're not sure what to do with it, try a double angle formula.
- For an expression for $\sin 3A$, write it as $\sin(2A + A)$ then use both compound and double angle formulae.
- When solving an equation of the form $\sin(a\theta + b) = k$, you may be tempted to expand using the compound angle formulae. Don't! Take an inverse sine to get $a\theta + b = \arcsin k$ and work from there.

Simplifying $a\sin\theta \pm b\cos\theta$ to $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$

For positive values of a and b,

$$\boldsymbol{a\sin\theta \pm b\cos\theta} \textbf{ can be expressed in the form } \boldsymbol{R\sin(\theta \pm \alpha)}$$

$$\boldsymbol{a\cos\theta \pm b\sin\theta} \textbf{ can be expressed in the form } \boldsymbol{R\cos(\theta \mp \alpha)}$$

with $R > 0$ and $0 < \alpha < 90°$ or $\frac{\pi}{2}$

The values of a, b, R and α are connected by

$$\boldsymbol{R^2 = a^2 + b^2}$$

$$\boldsymbol{R\cos\alpha = a} \textbf{ and } \boldsymbol{R\sin\alpha = b} \quad \Rightarrow \quad \boldsymbol{\alpha = \tan^{1}\left(\frac{b}{a}\right)}$$

Why this works

$R\sin(\theta \pm \alpha) = R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$ using the compound-angle formula

$R\sin(\theta \pm \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$ expanding the bracket

Since R and α are both constants, this can be written as:

$R\sin(\theta \pm \alpha) = a\sin\theta + b\cos\theta$ where $a = R\cos\alpha$ and $b = R\sin\alpha$

A similar derivation can be used for the $R\cos(\theta \pm \alpha)$ formula.

Example

Write $2\sin x + 2\sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$ where $R > 0$ and $0 < \alpha < 90°$ or $\frac{\pi}{2}$

First, write the appropriate compound-angle formula for reference:

$$R\sin(x + \alpha) = R(\sin x\cos\alpha + \cos x\sin\alpha)$$

Now the real work! Use Pythagoras to take out an appropriate factor:

$2\sin x + 2\sqrt{3}\cos x = 4\left(\frac{2}{4}\sin x + \frac{2\sqrt{3}}{4}\cos x\right)$ since $\sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$2\sin x + 2\sqrt{3}\cos x = 4\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

Match up the coefficients of sin and cos with the compound angle formula:

$\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \alpha = 60°$

We can now write the result:

$2\sin x + 2\sqrt{3}\cos x = 4\sin(x + 60°)$