AS Pure – Chapter 8 – Binomial Expansion

Factorials

For $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers, or positive integers), $\boldsymbol{n}!$ denotes "$\boldsymbol{n}$ **factorial**":

$$\boldsymbol{n! = n \times (n-1) \times (n-2) \times (n-3) \times \ \ldots \ \times 3 \times 2 \times 1}$$

Combinations

The number of ways of choosing r items from a group of n items (regardless of order) is written as ${}^nC_r$ or $\binom{n}{r}$.

This can be calculated as:

$$\boldsymbol{{}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}}$$

For example, ${}^5C_2 = \dfrac{5!}{2! \times 3!} = \dfrac{5\times4\times\cancel{3\times2\times1}}{2\times1\times\cancel{3\times2\times1}} = \dfrac{20}{2} = 10$

This means there are 10 ways of choosing 2 items from a group of 5 items.

Note that r can equal 0 here: there is always exactly 1 way of choosing zero items from a group of any size.

Pascal's Triangle

Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers of the next row:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

The rth entry in the nth row of Pascal's triangle is given by $\boldsymbol{{}^{n-1}C_{r-1}}$

This means the $(n+1)$th row of the triangle can be used to read off values of ${}^nC_r$ for reasonably small values of n.

Using the 5th row as an example:

$${}^4C_0 = 1$$

$${}^4C_1 = 4$$

$${}^4C_2 = 6$$

$${}^4C_3 = 4$$

$${}^4C_4 = 1$$

Binomials

A **binomial** is a polynomial which is the sum of two terms, which we often use enclosed in a set of brackets.

Examples include $(x+1)$, $(x+y)$, $(2x^2-3x)$ and (ax^m+by^n)

Binomial Expansion

The binomial expansion is:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n \qquad n \in \mathbb{N}$$

In the expansion of $(a+b)^n$, a general term is given by:

$$\boldsymbol{{}^nC_r\, a^{n-r}b^r} \quad \text{where } r = 0, 1, 2, \ldots, n$$

For example, in the expansion of $(5+2x)^4$, the five terms are given by:

$$\begin{aligned}
{}^4C_0\,(5)^4(2x)^0 &= (1)(625)(1) &&= 625\\
{}^4C_1\,(5)^3(2x)^1 &= (4)(125)(2x) &&= 1000x\\
{}^4C_2\,(5)^2(2x)^2 &= (6)(25)(4x^2) &&= 600x^2\\
{}^4C_3\,(5)^1(2x)^3 &= (4)(5)(8x^3) &&= 160x^3\\
{}^4C_4\,(5)^0(2x)^4 &= (1)(1)(16x^4) &&= 16x^4
\end{aligned}$$

So,

$$\boldsymbol{(5+2x)^4 = 625 + 1000x + 600x^2 + 160x^3 + 16x^4}$$

Notice that the ${}^nC_r$ values here (1, 4, 6, 4, 1) are the numbers in the 5th row of Pascal's triangle!

This is a useful time-saver with small values of n for which the relevant row of Pascal's triangle is quick to work out.

Binomial Approximations

If x is small, the first few terms of a binomial expansion can be used to find an approximation for a tricky calculation.

For example, substituting $x = 0.01$ into the first few terms of $(1+2x)^9$ gives an approximation for 1.02^9.

To work out the percentage error of the approximation, use your calculator to work out:

$$\frac{Approximation - True\ value}{True\ value} \times 100$$