AS Pure – Chapter 13 – Integration

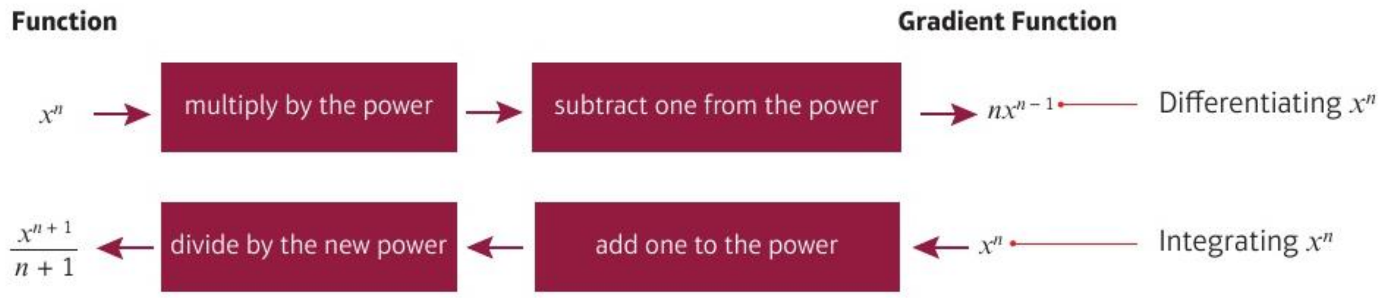
Integration is the opposite of **differentiation**. It allows us to find the general equation describing the shape of a curve if we know its gradient function. If we also know a point on the curve, we can find the exact equation.

More importantly, it can be used to find **areas under curves** and (for further mathematicians) volumes of 3D shapes.

Integrating Powers of x

When differentiating a power of x, we multiply by the power then reduce the power by one.

When integrating a power of x, we reverse this: increase the power by one then divide by the new power.



Be careful! When differentiating, any constants in the original equation don't appear in the gradient.

So when integrating we must add an arbitrary value (we use c) to represent the constants that *may* have been there.

This c is known as the **constant of integration**, and gives what is known as a **general solution** to the integral.

The process described above gives rules to find the equation of a polynomial if you know the gradient function:

If $\frac{dy}{dx} = x^n$, then $\boldsymbol{y = \frac{1}{n+1}x^{n+1} + c}$ If $f'(x) = x^n$, then $\boldsymbol{f(x) = \frac{1}{n+1}x^{n+1} + c}$ $\quad (n \neq -1)$

If $\frac{dy}{dx} = kx^n$, then $\boldsymbol{y = \frac{k}{n+1}x^{n+1} + c}$ If $f'(x) = kx^n$, then $\boldsymbol{f(x) = \frac{k}{n+1}x^{n+1} + c}$ $\quad (n \neq -1)$

Note that we can't use this rule for x^{-1}, as it would require division by zero. This will be addressed in Y13!

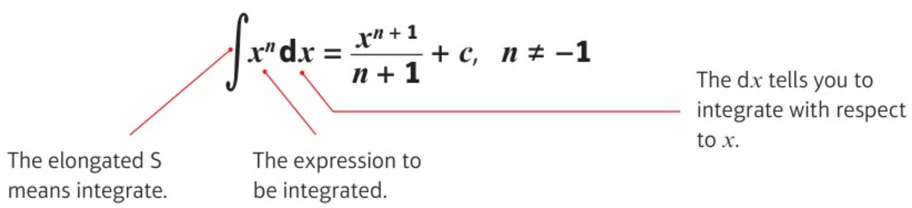
Notation for Integration

We use the symbol $\int$ to represent the process of integration:

$$\int f'(x)\,dx \;=\; f(x) + c$$

This is an **indefinite integral**, as the arbitrary constant c means there are technically an infinite number of solutions!

The process of integrating x^n can be written as follows:



When you are integrating polynomials, apply this rule separately to each term, like when differentiating:

$$\int [\,f(x) + g(x)\,]\,dx \;=\; \int f(x)\,dx \;+\; \int g(x)\,dx$$

Finding a Particular Solution

When integrating a gradient function, if we know a point on the original curve, we can find the value of c.

This turns the **general solution** of an integral (with a "$+c$" term) into a **particular solution**.

To find the **constant of integration** c, substitute the values of a point on the curve $(x\,,y)$ into the general solution.

This gives an equation which can be solved to find c.

Definite Integration

You can calculate an integral between two **limits**, or **bounds**. This is called a **definite integral**.

A definite integral usually produces a **value**, whereas an indefinite integral always produces a **function**.

If $f'(x)$ is the derivative of a function $f(x)$ for all values of x in the interval $[a\,,b]$, then the **definite integral** is:

$$\int_a^b f'(x)\,dx \;=\; [\,f(x)\,]_a^b \;=\; f(b)-f(a)$$

This is also known as a bounded integral. Here, a is the **lower bound** of the integral, and b is the **upper bound**.

Here are the steps for integrating the function $3x^2$ between the limits $x = 1$ and $x = 2$:

The limits of the integral are from $x = 1$ to $x = 2$.

Write the integral in [ ] brackets.

$$\int_1^2 3x^2\,dx = [x^3]_1^2$$

$$= (2^3) - (1^3)$$

Write this step in ( ) brackets.

Evaluate the integral at the upper limit.

$$= 8 - 1$$

Evaluate the integral at the lower limit.

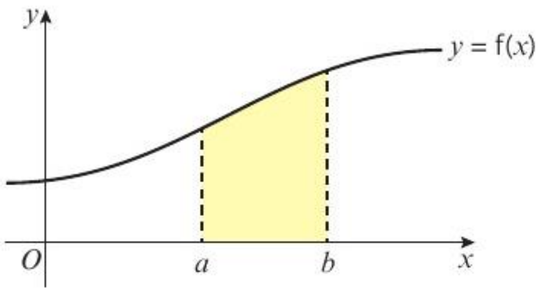
$$= 7$$

These definite integrals are hugely important as we can use them to find exact areas under curves.

Areas under Curves

The **area** between a **continuous positive curve**, the x-axis and the vertical lines $x = a$ and $x = b$ is given by:

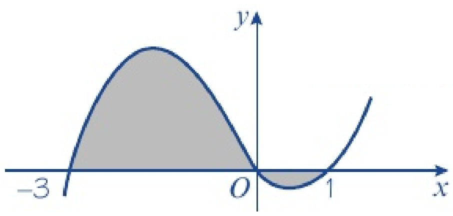
$$\textbf{Area} \;=\; \int_a^b y\,dx$$



where $y = f(x)$ is the equation of the curve. This is a **definite integral**, which gives a value for the area.

When the area bounded by the curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative value.

If a curve has areas above and below the x-axis, find the roots of the equation and consider the areas separately.



Here, calculating $\int_{-3}^{1} y\,dx$ would give the difference between the two areas (as the negative area would cancel some of the positive area, not add to it).

Instead, we need to calculate:

$$\text{Total Area} \;=\; \left|\int_{-3}^{0} y\,dx\right| + \left|\int_{0}^{1} y\,dx\right|$$

Note the modulus signs around each interval.

These tell us to take a positive value for each area, even if the integral is negative.