A2 Pure – Chapter 11 – Integration

Standard Integrals

You need to **know** these standard integrals, which are simply the inverses of known derivatives:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

$$\int e^x \, dx = e^x + c \qquad\qquad \int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int \sin x \, dx = -\cos x + c \qquad\qquad \int \cos x \, dx = \sin x + c$$

More standard integrals which you may need to use are given in the formula book, such as:

$$\int \sec^2 x \, dx = \tan x + c$$

Using Trigonometric Identities

Trigonometric identities can be used to re-write some integrals into a form which matches standard results.

For example, we don't know how to integrate $\tan^2 x$, but we do know that $\tan^2 x + 1 \equiv \sec^2 x$.

Using this identity, we can write that $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c$ as the integral of $\sec^2 x$ is a standard result given in the formula book.

Using trigonometric identities for this takes practice, so don't worry if you struggle to spot them at first! Look at the standard derivatives in the formula book as well as the integrals.

For example, since you are given $\frac{d}{dx}(\sec x) = \sec x \tan x$, then we know that $\int \sec x \tan x \, dx = \sec x + c$

Integration by Reversing the Chain Rule

There are two similar cases in which the chain rule for differentiation is easily reversed.

If a function can be written in either of these forms you can integrate by reversing the chain rule for differentiation:

$\frac{kf'(x)}{f(x)}$ (the numerator is the derivative of the denominator)

$kf'(x)\left(f(x)\right)^n$ (the function at the front is the derivative of the function in the power)

To integrate expressions of the form $\int k \frac{f'(x)}{f(x)} \, dx$

try $\ln|f(x)|$ as a possible solution, differentiate using the chain rule to check, and adjust any constant if required.

To integrate expressions of the form $\int k \, f'(x) \left(f(x)\right)^n \, dx$

try $\left(f(x)\right)^{n+1}$ as a possible solution, differentiate using the chain rule to check, and adjust any constant if required.

This can also be done by **inspection**:

"increase the power by one, divide by the new power, and divide by the derivative of the function in the power"

When using inspection, it can help to bring any constant in the integral to the front to keep it out of the way!

As always, and especially with trigonometric functions, be sure to make use of the standard integrals given in the formula book to help you identify when a function is in a suitable form for this technique to be used.

Integration by Substitution

Sometimes you can simplify an integral by changing the variable. This process is similar to the chain rule for differentiation, and is called integration by substitution. In the exam you may be told which substitution to use.

Otherwise, look for integrals of $\frac{kf'(x)}{f(x)}$ and $kf'(x)\big(f(x)\big)^n$, where you can use $\boldsymbol{u = f(x)}$ as the substitution.

If you are not confident reversing the chain rule, substitution can always be used instead!

Another common type of substitution is for fractions with a polynomial numerator and linear denominator, such as

$$\int \frac{f(x)}{ax+b}\,dx$$

Here, you would use the substitution $u = ax + b$ and write the numerator in terms of u as well, to get

$$\int \frac{g(u)}{u}\,dx$$

As $g(u)$ will also be a polynomial, the fraction can now be separated out and powers of u simplified then integrated.

Partial Fractions

You may be able to integrate algebraic fractions by using partial fractions into write as simpler fractions which can be integrated separately. These simpler fractions will have constant numerators and linear denominators, which integrate to give natural logarithms.

For example,

$$\int \frac{x-5}{x^2-x-2}\,dx = \int \frac{x-5}{(x+1)(x-2)}\,dx \qquad \text{(Factorise the denominator)}$$

$$= \int \frac{2}{x+1}\,dx - \int \frac{1}{x-2}\,dx \qquad \text{(Using partial fractions, see Chapter 1 for method)}$$

$$= 2\ln|x+1| - \ln|x-2| + c \qquad \text{(Using inspection, reserve chain rule or substitution)}$$

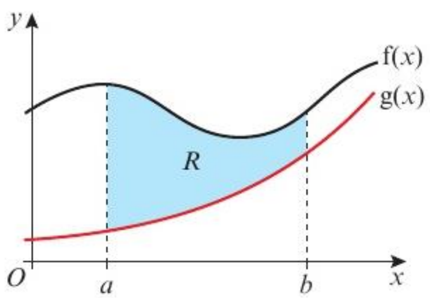
$$= \ln\left|\frac{(x+1)^2}{x-2}\right| + c \qquad \text{(Optional, using log laws to combine the logarithms)}$$

Areas between Curves

You saw in Y12 how to use definite integrals to find the area between a curve and the x-axis.

You can also find the area between two curves by **integrating the difference between the functions** or by finding the difference between the two integrals:

$$\boldsymbol{Area\ of\ R} = \int_a^b [\,\boldsymbol{f(x) - g(x)}\,]\,\boldsymbol{dx} = \int_a^b \boldsymbol{f(x)} - \int_a^b \boldsymbol{g(x)}$$



This only works if the two curves **do not intersect** in the range $[a\,, b]$

Otherwise you should do separate calculations for each section.

Integration by Parts

You have already met the product rule for differentiation:

$$(uv)' = uv' + u'v \qquad \text{also written as} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This can be rearranged:

$$(uv)' = uv' + u'v$$

$$uv' = (uv)' - u'v$$

Integrate each term on both sides:

$$\int uv'\,dx = \int (uv)'\,dx - \int u'v\,dx$$

$$\boldsymbol{\int uv'\,dx = uv - \int u'v\,dx}$$

We now have a formula which allows us to integrate a product of two functions, u and v', which can't be integrated directly. This works as long as u' and v both exist and the product $u'v$ can be integrated.

This method is known as **integration by parts**, and the formula is given in the formula book in $\frac{d}{dx}$ form:

$$\boldsymbol{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

For example, let's integrate the function $2x\sin x$

One part of the function will be our u, and the other will be v'. I will choose $u = 2x$ and $v' = \sin x$.

Write out u, u', v and v'. We need to quickly differentiate $2x$ and integrate $\sin x$ to complete this:

$$u = 2x \qquad v = -\cos x$$

$$u' = 2 \qquad v' = \sin x$$

Then set up the integral using the formula:

$\int uv'\,dx = uv - \int u'v\,dx$ → $\int 2x\sin x\,dx = (2x)(-\cos x) - \int 2(-\cos x)\,dx$

Tidy this up a bit: → $\int 2x\sin x\,dx = -2x\cos x + \int 2\cos x\,dx$

The integral on the RHS is trivial → $\int 2x\sin x\,dx = -2x\cos x + 2\sin x + c$

Done! Don't forget the '$+c$' for this indefinite integral.

You have to choose which part of the original function will be u and which will be v'. You can just decide this at random, and see if the integration works. If not, swap them round and try again. But a few basic tips can help you get this choice right first time, based on the fact that u needs to be differentiated and v' needs to be integrated:

1. When the expression involves $\ln x$, always set this to be u as it differentiates easily.
2. Powers of x get simpler when differentiated, so set these as u if possible.
3. e^x, $\sin x$ and $\cos x$ are just as easily integrated as differentiated, so they are often set as v'

Solving Differential Equations

A differential equation is a mathematical equation that relates some function with its derivatives.

In practical applications, the functions usually represent physical quantities, the derivatives represent their **rates of change**, and the equation defines a relationship between the two.

In A-level Maths you will solve **first order** differential equations – those containing a first derivative such as $\frac{dy}{dx}$ but no higher derivatives. Second order differential equations are only covered in Further Maths. Sorry.

Solutions to differential equations can be found by integration. You have already seen simple examples of these in AS Maths when you were asked to integrate gradient functions, for example:

First order differential equation: $\frac{dy}{dx} = 12x^2 - 1$

General solution: $y = 4x^3 - x + c$ or $y = x(2x+1)(2x-1) + c$

This gives the **general solution** to the differential equation (with a **constant of integration**) and represents a family of solutions, all with different constants and all satisfying the original equation.

To find a **particular solution**, you need to be given a point on the curve so you can calculate the value of the constant. This is sometimes called a **boundary condition**.

Integration can be used to solve more complex differential equations by **separating the variables**:

When $\boldsymbol{\frac{dy}{dx} = f(x)g(y)}$, you can rearrange this to give

$$\boldsymbol{\int \frac{1}{g(y)}\, dy = \int f(x)\, dx}$$

These often give natural logarithms of y (or some function of y) on the left-hand side. These can be cancelled out by taking natural exponents (powers of e) on both sides.

A common application of differential equations is to model real-life situations, such as radioactive decay or population growth. These often give natural logarithms of y (or some function of y) on the left-hand side. By taking natural exponents (powers of e) on both sides, a common general solution is

$$y = e^{f(x) + c}$$

Note that the constant of integration is in the exponent. This can be rewritten using index laws as follows:

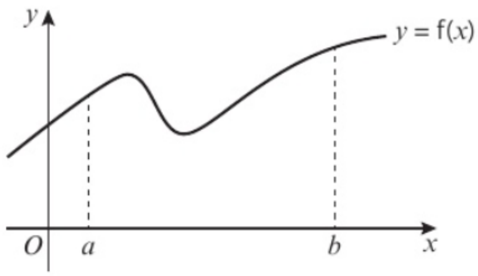
$$e^{f(x)+c} = e^{f(x)} \times e^{c} = Ae^{f(x)} \quad \Rightarrow \quad \boldsymbol{y = Ae^{f(x)}}$$

where A is a new constant, often an initial value for y, which can be found using the boundary conditions.

The Trapezium Rule

If you cannot integrate a function algebraically, you can use a numerical method to approximate the area beneath a curve. One such method is the **trapezium rule**.

Consider the curve $y = f(x)$:

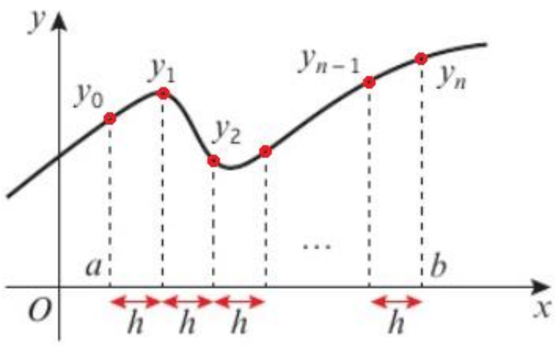


To approximate the area given by $\int_a^b y\,dx$, you can divide the area into n equal strips of width h, where $h = \frac{b-a}{n}$.

The area of these strips can be estimated by considering each as a trapezium of 'height' h.

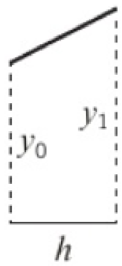
The sum of the areas of the n trapezia gives us an approximation for the integral.

To find the area of each trapezium, we also need the height of the left and right boundaries for each strip. Calculate the value of y for each value of x on the boundary of one of the strips. These values can be labelled $y_0, y_1, \dots, y_n$:



Finally, create the trapezia by drawing chords for y_0 to y_1, y_1 to y_2, etc.

You know from GCSE maths that the area of the trapezium like this is given by $\frac{1}{2}(y_0 + y_1)h$



So the area under the curve is given by the sum of the trapezia:

$$\int_a^b y\,dx = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \dots + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h$$

Factorising gives:

$$\int_a^b y\,dx = \frac{1}{2}h\,[(y_0 + y_1) + (y_1 + y_2) + \dots + (y_{n-2} + y_{n-1}) + (y_{n-1} + y_n)]$$

Collecting terms:

$$\int_a^b y\,dx = \frac{1}{2}h\,[\,y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n\,]$$

This is the result given in the formula book:

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The more strips you use, the more accurate the estimate from the trapezium rule will be.

For convex curves, the trapezium rule will give an overestimate, for concave curves it will be an underestimate.