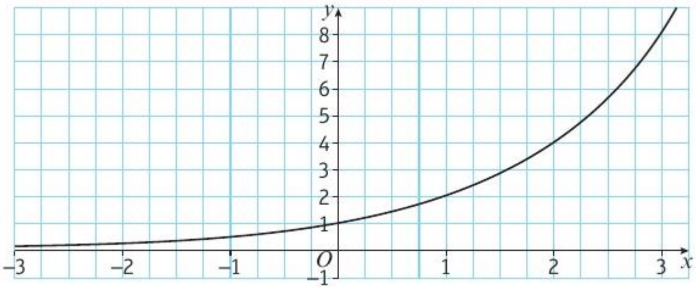
AS Pure – Chapter 14 – Exponentials and Logarithms

Exponentials

Functions of the form $f(x) = a^x$, where a is a constant, are called exponential functions.

You should become familiar with these functions and the shapes of their graphs, for example $y = 2^x$:



The x-axis is an asymptote to the curve

The y-intercept is at $(0, 1)$, as $2^0 = 1$.

In fact, any graph of the form $y = a^x$ will pass through $(0, 1)$.

Euler's number, e

Exponential graphs of the form $y = a^x$ have the special property that the graphs of their gradient functions are stretches of the graphs themselves. In one special case, where a is roughly 2.71828, the gradient function is identical to the original function. This value is represented by the letter $\boldsymbol{e}$, and (like π) it is an **irrational number**.

$$\boldsymbol{e \approx 2.71828}$$

For all real values of x, if $y = e^x$ then $\dfrac{dy}{dx} = e^x$ **For all real values of x, if $y = e^{kx}$ then $\dfrac{dy}{dx} = ke^{kx}$**

You can use e^x to model situations such as population growth, where the rate of **increase** is proportional to the size of the population at any given moment. Similarly, e^{-x} can be used to model situations such as radioactive decay, where the rate of **decrease** is proportional to the number of atoms remaining.

These models take the general form $y = Ae^{kx}$, where A and k are constants.

Logarithms

The inverse of an exponential is called a **logarithm**, often abbreviated as **log**.

A relationship which is expressed using an exponent can also be written in terms of logarithms:

$$\boldsymbol{\log_a n = x} \quad \textbf{is equivalent to} \quad \boldsymbol{a^x = n}$$

a is the **base** of the logarithm, and $\log_a n$ essentially means "the power to which a is raised to give the result n".

For example, $\log_{10} 1000 = 3$, because $10^3 = 1000$

Laws of Logarithms

Expressions involving logarithms can often be rearranged and simplified using the laws of logarithms, which are basically the laws of indices from GCSE Maths rewritten in logarithmic form:

$$\boldsymbol{\log_a x + \log_a y = \log_a xy} \qquad \boldsymbol{\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)} \qquad \boldsymbol{\log_a x^k = k\log_a x}$$

Be careful! There aren't laws for multiplying and dividing logarithms, don't invent any!

You also need to know these special cases:

$$\boldsymbol{\log_a \left(\frac{1}{x}\right) = \log_a(x^{-1}) = -\log_a x}$$

$$\boldsymbol{\log_a a = 1} \ \ (since\ a^1 = a) \quad \text{and} \quad \boldsymbol{\log_a 1 = 0} \ \ (since\ a^0 = 1) \qquad \boldsymbol{a > 0\,,\ a \neq 1}$$

Using Logarithms

You can solve equations with unknown powers by taking logarithms of both sides. For example:

$2^x = 50$	
$\log(2^x) = \log(50)$	Taking logs of both sides
$x \log 2 = \log 50$	Using the power law
$x = \frac{\log 50}{\log 2}$	This is considered an exact value for x

You can use your calculator to get a value to a particular degree of accuracy, say 3 significant figures.

Note that the base of the logarithms wasn't stated in this example. In fact, any base would give the correct result!

Generally, if the base isn't stated we assume base 10 is being used – the 'log' button on a calculator will use base 10.

Natural Logarithms

Logarithms with base e are so frequently used that they have their own notation: $\log_e x = \ln x$

The function $\ln x$ is the inverse function of e^x, and their graphs are reflections in the line $y = x$.

The fact they are inverses can allow us to solve equations involving powers of e and natural logarithms:

$$\boldsymbol{e^{\ln x} = x} \qquad \text{and} \qquad \boldsymbol{\ln(e^x) = x}$$

Logarithmic Graphs

There are two particular cases where non-linear relationships can be rewritten in terms of logarithms in such a way as to create a straight-line graph: $y = ax^n$ and $y = ab^x$, where a, b and n are constants.

From these graphs, the gradients and intercepts can give us useful information about the relationships.

Note the key difference: in the first case, the variable x is the base, and in the second case x is the power.

Case One: $y = ax^n$	**Case Two: $y = ab^x$**	
$y = ax^n$	$y = ab^x$	Start with the assumed relationship
$\log y = \log(ax^n)$	$\log y = \log(ab^x)$	Take logs of both sides
$\log y = \log a + \log(x^n)$	$\log y = \log a + \log(b^x)$	Separate using log laws
$\log y = \log a + n \log x$	$\log y = \log a + x \log b$	Use log laws on the power

Compare these results to the equation of a straight line $y = mx + c$.

If your data fits the model $y = ax^n$, you will get a straight line if you plot $\log y$ against $\log x$

The gradient will be n and the vertical intercept will be $\log a$

If your data fits the model $y = ab^x$, you will get a straight line if you plot $\log y$ against x

The gradient will be $\log b$ and the vertical intercept will be $\log a$

If you suspect your data may fit an exponential model, try both cases and see which one gives a straight-line graph.